WORLD THEORY

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Abstract. In this paper a general mathematical model of the World will be constructed. I will show that a number of important theories in Physics are particularizations of the World Theory presented here. In particular, the worlds described by the Classical Mechanics, the Theory of Relativity and the Quantum Mechanics are examples of worlds according to this definition, but also some theories attempting to unify gravity and QM, like String Theory. This mathematical model is not a Unified Theory of Physics, it will not try to be a union of all the results. By contrary, it tries to keep only what is common and general to most of these theories. Special attention will be payed to the space, time, matter, and the physical laws.

What do we know about the laws governing the Universe? What are the most general assumptions one can make about the Physical World? Each theory in Physics and each philosophical system came with its own vision trying to describe or explain the World, at least partially. In the following, I will try to keep the essential, and to establish a mathematical context, for all these visions. The purpose of this distillation is to provide a mathematical common background to both physical and metaphysical discussions about the various theories of the World.

The mathematical object named World is defined using the locally homogeneous sheaves and sheaf selection which are introduced in the appendices.

Contents

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1. **Introduction**

This article starts by introducing a mathematical structure, named *world*, that captures the main aspects of a theory in Physics (section §2). This structure can be particularized such that we obtain some of the most important theories in Physics. Some of the mathematical notions used for this purpose are introduces in the appendices.

The world is defined as a mathematical structure containing the spacetime, which in general is a topological space, and the physical laws, expressed as a sheaf over the spacetime. Considerations of symmetry suggest that, in general, the law sheaf is locally homogeneous.

The causal structure is another construction, which imposes some restrictions on the way we can define the time. A *causal world* is defined as a world endowed with a causal structure (section §3).

The definition of a world can be generalized, to fit the necessities of the most non-classical theories in Physics. Some mechanisms related to the construction of a theory in Physics can be revealed through the World Theory in section §5. It is discussed how complexity of a theory affects its beauty, comprehensibility, and effectiveness.

In the Appendices, I introduced the locally homogeneous topological spaces as topological spaces such that any pair of points x, y admits open neighborhoods U, V and a local homeomorphism mapping $f : U \to V$ such that $f(x) = y$. The locally homogeneous presheaves are characterized by the condition that they admit transitive pseudogroups of transformations (the transformations are also required to be homeomorphisms).

I start from the observation that a PDE problem has two sets of data: the PDE equation (or system of equations), and the initial or boundary data. I extended these ideas to general sheaves, so that a homogeneous sheaf corresponds to the PDE equation, and a subsheaf selection to the initial/boundary data. I posed two types of problems. The first one refers to finding the sections of the locally homogeneous sheaf (corresponding to the PDE), subject to the subsheaf selection (corresponding to the initial or boundary conditions). The second refers to finding a minimal homogeneous sheaf containing a given subsheaf, or admitting a given section. After that, I expressed the concepts introduced here in more general, topos theoretical settings.

2. **World Theory and Physics**

2.1. **The mathematical description of the World**

Science is the study of the rules governing the world. The way of science is to propose hypotheses about what the rules are, and to test them through their consequences. It is clear that the logic plays a very important role, in deriving the consequences, in developing the explanations, in checking the logical consistency of each theory. Science assumes that the logical structure of the theories must be flawless. This may allow us to express as axioms the fundamental principles, and therefore to provide a mathematical description of the world. Many physicists strongly believe that the laws of Nature are expressible in a mathematical form ([43, 41]). In fact, the very existence of a science like Physics is a proof that they believe so. Of course, at any moment of the history of science there may exist great philosophers which express doubts about the necessity that the Nature obeys a mathematical law.

It is not the purpose of this paper to solve this metaphysical dilemma. Maybe the Nature can be entirely described mathematically, or maybe only partially. I will only conjecture that there exist a part of the Nature that can be described mathematically. We are concerned only with that part of the Nature that accepts a logical and mathematical description. There are some hints that we can use Mathematics to describe, at least partially, the Nature. Physics proved itself to be successful in identifying such parts of the Nature, and describing them mathematically with an impressive degree of accuracy. Our claim is that the Nature can be expressed mathematically at least partially. It is this part we will discuss in the following, and I will model it by a mathematical structure named *world*.

More details may be required about what a mathematical description of the world can do. Suppose we know a set of axioms describing (at least partially) the laws of the real world. The axioms will represent mathematically and logically the relations between various objects of the world. For example, they will tell, considering that the space in the Newtonian Mechanics is Euclidean, that two points determine a unique line. They will tell nothing about what the points are, or what the line is. The formalist point of view saids that there is nothing beyond the relations described by the axioms, there is no need for a "physical" interpretation. Our view will be that, even if the points or lines have a physical meaning, the mathematical description will ignore it, and it will keep only the relation that can be expressed mathematically and described by axioms. What can be expressed mathematically, become part of the axiomatic theory. Any knowledge about *what* the object are is meaningless, except when the objects of a theory can be defined in terms of other objects of that theory. An "outside" explanation, even if it may exist, will not be considered as part of the theory. The only possibility to account for the "outside" objects or meanings will be by extending the theory. Even in this case, there will be in the theory fundamental objects that cannot be defined in terms of more fundamental ones. This is why an axiomatic theory describes only the relations between the objects, and any model (in the sense of the Model Theory) of that theory will not be part of the theory itself. Yet, any theorem deduced in the axiomatic theory will apply to the model too.

The process of abstraction and formalization will allow us to focus on the elements of the theory, and to ignore what is "outside", or the question if there is something "outside". This does not make any implication about the existence of the "outside"; from the theory's point of view it simply makes no sense to discuss about it.

In conclusion, I don't want to imply something about the existence of a complete mathematical description of the world. Also, I don't want to imply that the question about the nature of things, the personal metaphysical or philosophical views of the scientists, the inner representations of the world, are unimportant. I only want to discuss about what is expressible in terms of mathematics, and to ignore for the moment the rest and the question whether there exist something that escapes to any possible attempt of mathematical modeling.

2.2. **Spacetime**

Initially, the space has been considered to be a three-dimensional Euclidean space. The background of the physical phenomena, in the Newtonian Mechanics, was a direct product of a 3-dimensional Euclidean space and the time axis $\mathbb{R}$. The Special Theory of Relativity merged the time with the space and obtained a four-dimensional spacetime with Lorentz metric. The General Theory of Relativity allowed the spacetime to be curved, making it into a differentiable manifold with Lorentz metric. As a consequence of the curvature, the gravitation is interpreted as a geometric property of the spacetime. But there are material objects that cannot be explained (at least so far) by the spacetime geometry exclusively.

We accept that there is a spacetime – as an arena of all the phenomena. Of course, this arena can be not only a background, but an active participant to the phenomena, as in the General Theory of Relativity. We put as a first axiom that:

Principle 1. There exists a topological manifold named the *spacetime*.

Let's denote this spacetime manifold by $\mathcal{S}$.

Remark 2.1. We don't make here assumptions about the (topological, so far) dimension of $\mathcal{S}$, because we want to allow all the theories, no matter how many different dimensions they need for the spacetime. There is another reason for maintaining the generality: we will want to apply the formalism developed in this paper to phase or state spaces as well. In general it is enough to consider the spacetime as being a topological manifold, but a more general approach will be to consider it a locally homogeneous topological space (see Definition A.15 in the Appendices):

Principle 1'. There exists a locally homogeneous topological space named the *spacetime*.

Remark 2.2. It may seem that is a limitation to allow the spacetime only to be "continuous". The requirement that the spacetime be described by a topological space is not such a drastic constrain, because we can always consider it endowed with the *discrete topology*.

Also we want to allow the possibility that the spacetime is more general than a topological space, because there are some modern theories that require this. Starting with the observation that a topological space can be regarded as the category of its open sets, together with the inclusion maps as morphisms, we can make generalizations using the Category Theory. For example, the spacetime may be a locally homogeneous locale (see Definition C.11 in the Appendices):

Principle 1". There exists a locally homogeneous locale named the *spacetime*.

The reason for requiring the local homogeneity condition is that, in most theories, we believe that the spacetime looks similar in different points. Of course, especially when a theory assumes discreteness of the spacetime, it may be possible that the local homogeneity is not respected. For such situations, the first principle can be generalized even more, by eliminating the condition of local homogeneity, and even more by considering the spacetime as being a more general category. I would like to consider open the problem of the best condition for the spacetime $\mathcal{S}$.

We consider at the beginning the spacetime as a unity, although in some theories it can be regarded as a product of a space and a time, or time can be regarded as a fiber bundle, with the space being the fiber. We will discuss later the way spacetime splits in space and time.

2.3. Matter

What about the matter on the spacetime? In general, the matter fields are scalar, pseudoscalar, vectorial, pseudovectorial, tensorial or spinorial quantities. The natural way to see the matter fields is as sections in vector bundles. The scalar, vector, tensor and spinor fields are in fact functions valued in vector spaces, and the natural mathematical object that allows us to deal with such quantities is the notion of vector bundle. The matter fields will be sections of vector bundles, or more generally, sections of fiber bundles, with the spacetime as base manifold.

But, as far as we know, there exists several different kinds of matter fields, maybe it is better to have several bundles over the spacetime. We simply consider at each point of the spacetime the Cartesian product of all the fibers over that point (if the bundles are vector bundles, the Cartesian product will become the direct sum). Consequently, we consider a single bundle over the spacetime without losing in generality. In Quantum Mechanics or theories extending it, we need to consider tensor products of sections. In order to fit better these tensor products in a vector bundle, we can replace the state space with its purified version. All the matter fields can be incorporated in a unique matter field which is a section of the matter bundle.

We then consider that the matter is a section of a bundle over the spacetime:

Principle 2. There exists a fiber bundle over the *spacetime*, named the *matter bundle*.

We denote the matter bundle over the spacetime $\mathcal{S}$ by $\mathcal{M} \xrightarrow{\sigma} \mathcal{S}$, where σ is the canonical projection. $\mathcal{M} \xrightarrow{\sigma} \mathcal{S}$ may be a vector bundle, or just a topological bundle, or, more generally, it may be a functor to the spacetime. The sheaf of the sections of this bundle is named the *matter sheaf*. The matter field is a global section of the matter sheaf.

The main reason for considering sheaves instead of bundles is that some (in fact, most) of the physical laws are local[1]. In general, they are local and independent of the point of spacetime, and this is why we use sheaves, instead of simply using bundles. But of course, any bundle has associated a sheaf of local sections, and, reciprocally, any sheaf over a topological space can be made into an étale bundle, such that the sections in that sheaf are local sections of the étale bundle.

A more general approach is to start directly with the matter sheaf (see Definition A.19 in the Appendices):

Principle 2'. There exists a totally homogeneous sheaf over the *spacetime*, named the *matter sheaf*.

Of course, we can also renounce at the condition of total homogeneity, and impose instead only local homogeneity.

2.4. **Physical laws**

All the possible sections of the matter bundle form a sheaf over the spacetime, but not all are allowed by the physical laws. In some theories we reduce the sheaf by accepting only the solutions of the mathematical equations (in general partial differential equations – PDE) which express the physical laws. If the matter fields are solutions to PDE, they form a subsheaf of the sheaf of sections of the matter bundle.

Because we consider all the matter fields combined into a single one, the equations will also be combined in a single equation, which will describe the interactions between the various matter fields. This unique equation is simply a combination of all the equations that describe the physical laws for all the matter fields. It is not the "Unified Theory of Physics", because any set of equations can be "unified" in this manner. Of course, I will expect that the "Unified Theory" will have this form, but the component "sub-equations" must be parts of the "unified equation" in a more natural way.

For generality purposes, we don't refer to the equations, but rather to the subsheaf consisting in matter local fields admitted by the physical laws. This will allow us to speak generally about the physical laws without referring to PDE or other types of equations describing the physical laws. There are discrete theories, where PDE cannot express the laws.

Principle 3. There exists a locally homogeneous subsheaf of the matter sheaf, named the *physical laws sheaf* or simply the *law sheaf*.

We denote the law sheaf by Λ. The law sheaf is in general taken to be locally homogeneous, because physical laws are independent of space and time. Of course, the

[1]From topological point of view, most of the physical laws are local, but the physical solutions have to be global.

local homogeneity is defined only when the spacetime is locally homogeneous. If this is not the case, we can replace the local homogeneity condition for the law sheaf with the condition of local semi-homogeneity (see Definition A.21), which is defined for general topologies.

The set of global sections of the law sheaf is named the *set of solutions*, in analogy to the set of solutions of a PDE.

The matter field should not only be a section of the matter bundle, but also a section of the law sheaf, because it has to obey the Physical Laws:

Principle 4. There exists a solution (global section of the *law sheaf*), named the *matter field*.

The solution can be determined by a set of conditions. In the theory of PDE for example is the initial condition that determine a solution of the PDE. More generally, instead of initial conditions we have a *selection* of the matter field in the law sheaf (see Definition B.3).

Let's denote this section by $\mu \in \Lambda(\mathcal{S})$. Of course, $\mu \in \Gamma(\mathcal{S}, \mathcal{M})$ also holds.

Remark 2.3. The matter sheaf is an invariant of the topological structure of $\mathcal{S}$, while the law sheaf Λ is an invariant of the pseudogroup $\mathcal{T}(\Lambda)$ it generates (see Definition A.12). $\mathcal{T}(\Lambda)$ is a more restrictive structure on $\mathcal{S}$, being a subpseudogroup of the pseudogroup $\mathcal{T}(\mathcal{S})$. For example, $\mathcal{T}(\Lambda)$ can be the pseudogroup providing a differential structure on $\mathcal{S}$, or something even stronger such as the pseudogroup of transformations of the solutions of a PDE. Why we distinguish the matter sheaf from the law sheaf, when we could have specified only the law sheaf? Mainly because, as the example of the PDE suggests, we need a stable reference. $\mathcal{S}$ being a topological space, its topological structure provides the pseudogroup $\mathcal{T}(\mathcal{S})$, and it is natural to consider a $\mathcal{T}(\mathcal{S})$-invariant sheaf, and then to refine it in order to obtain the law sheaf.

2.5. **Symmetries and the levels of the physical laws**

If we refer to our particular example of physical law, the one of PDE, we can see that in fact, the law provides just the evolution equation, having an infinity of solutions. In order to specify a solution one need a condition at a given moment $t_0 = 0$. In the case of most equations of the Physics, specifying the equation of evolution and the initial conditions gives a unique solution. In the following, we will exemplify the way we choose the law sheaf, applying considerations of symmetry, which are comprised in the condition of local homogeneity of the law sheaf.

Example 2.4. For example, let's consider a material point moving with a given constant acceleration $\mathbf{a}$ in the Euclidean plane $\mathbb{R}^2$. Let's suppose that the evolution is described by the equation

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \mathbf{a} \tag{1}$$

which admits solutions of the form

$$\mathbf{x}(t) = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{x}_0, \tag{2}$$

where $\mathbf{a}$, $\mathbf{v}_0$, $\mathbf{x}_0$ and $\mathbf{x}$ are vectors in $\mathbb{R}^2$, $\mathbf{a}$ is given, $\mathbf{v}_0$ and $\mathbf{x}_0$ are free parameters. Let's denote the space of functions of this form by Λ, it defines a sheaf. The spacetime is given by the real line $\mathbb{R}$ representing the time, the matter bundle is given by the bundle $\mathbb{R}\times\mathbb{R}^2 \stackrel{\pi_1}{\to} \mathbb{R}$, and the sheaf Λ is a subsheaf of the sheaf of sections $\Gamma(\mathbb{R}\times\mathbb{R}^2 \stackrel{\pi_1}{\to} \mathbb{R})$. If we like, we can organize the data so that the law sheaf is defined over the spacetime $\mathbb{R}\times\mathbb{R}^2$. In this case the law sheaf associates to each open subset of $\mathbb{R}\times\mathbb{R}^2$ the characteristic function of the graph of the functions represented by the equation (2). We will describe this idea in the example 2.20.

Knowing

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{3}$$

for an initial moment $t_0 = 0$ gives us an initial condition. This condition does not establish a unique solution, but a subspace of the general solution space, which is of the form:

$$\mathbf{x}(t) = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{x}_0,$$

where the only free parameter is now $\mathbf{v}_0$. These solutions defines a subsheaf Λ' of the sheaf Λ of all solutions. We also need to know $\mathbf{v}_0$, the initial speed – the first derivative of $\mathbf{x}(t)$ at the moment $t_0 = 0$. Let's write this second initial condition:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{t}}(t_0) = \mathbf{v}_0. \tag{4}$$

This set of two conditions reduces the solution space to a space containing a unique element, which is just the solution, and defines a sheaf Λ''. One can say that the solution is given in term of three equations – the evolution equation, the condition for the initial position and the condition for the initial speed. What is the law sheaf for this case? It is natural to assume that Λ is the law sheaf. But the condition for the initial position defines a subsheaf Λ' in this sheaf, isn't it possible to consider this sheaf instead? The answer is yes, but our common sense tells us that Λ is more appropriate, because it is independent of the value $\mathbf{x}$ may take at a particular moment $t_0 = 0$. Of course, Λ depends on the acceleration $\mathbf{a}$, but $\mathbf{a}$ is presumed independent of t.

The example above showed that when we consider the physical laws expressed as equations of evolution, it appears to exist a natural way to distinguish the law sheaf from the subsheaves determined by particular initial conditions. The distinction is made by the condition that all the coefficients in the equation be independent of time and space. But $\mathbf{v}_0$ and $\mathbf{x}_0$ are constants as well, hence each one of the the sheaves Λ, Λ' and Λ'' can be considered as law sheaf. Still, why it seems more natural to choose Λ? Let's apply a transform to the time, $t \mapsto t+\theta$. The equation (2) is transformed in

$$\mathbf{x}(t) = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_1 t + \mathbf{x}_1, \tag{5}$$

where $\mathbf{x}_1 = \frac{1}{2}\mathbf{a}\theta^2 + \mathbf{v}_0\theta + \mathbf{x}_0$ and $\mathbf{v}_1 = \frac{1}{2}\mathbf{a}\theta + \mathbf{v}_0$. One can observe that only $\mathbf{a}$ is independent of the time coordinate, while the coefficients $\mathbf{v}_0$ and $\mathbf{x}_0$ are not. We can extract the following rule: if the law sheaf contains the function $\mathbf{x}(t)$, it will also contain $\mathbf{x}(t+\theta)$. This means that the law sheaf is invariant to time translations. More generally, we want the physical laws to be independent of space and time, and this is why we prefer a locally homogeneous law sheaf.

The invariance properties are used to characterize the properties of the law sheaves. In general the law sheaf can be defined in more than one way. We want the law sheaf to describe best the systems we are studying. We want that the description to be general – for example to be independent of the particular time and position, or of the initial conditions. In the same time, we want the description to be not too generic, otherwise it will not be useful. For instance, in the same example 2.4 we could define the law sheaf as the sheaf of all continuous functions $x : \mathbb{R} \to \mathbb{R}^2$, but this definition is too generic, and does not say much about the accelerated motion. This is why we preferred the equation (2).

The choice of the law sheaf depends on the intended level of abstraction – in the example above, if we just want to study the continuous motions, the sheaf of continuous functions is good enough. If we want to study the generic accelerated motions, we can consider a sheaf of accelerated motions with all the possible accelerations. If we want to study a particular example, we can consider the sheaf Λ'' which contains a unique solution.

In general in Physics, the laws are independent of the initial conditions, but there are theories which study precisely the initial conditions – such as the theories of the origin of the physical world. There exists also laws of Physics which are the same at any moment and position, but are dependent on the initial conditions of the Universe. For example the Second Law of Thermodynamics is valid everywhere (although its validity is only statistical), but it depends on the initial low level of entropy of the Universe. Other theories explain some properties of the particles and their interactions by a spontaneous symmetry breaking occurred in the early ages of the Universe.

In most cases, the physical laws expressed by the law sheaf have some invariance properties. Sometimes the law sheaf is invariant to groups or pseudogroups of transformations of the underlying topological manifold which is the spacetime. This is the case of the Newtonian Physics, whose laws are invariant to global transformations which form the Galilei group, and with the Special Theory of Relativity, whose invariance group is the Poincaré group. Some equations, like Maxwell's, are invariants of a larger group – the conformal group $O(2,4)$. Spinor field equations in Special Relativity are invariants of the group $SL(2,\mathbb{C})$. The General Theory of Relativity states that the laws should be invariated by a pseudogroup of transformation – the transition functions of the semi-Riemannian manifold representing the spacetime. Transformations of the fibers also can leave the form of physical equations unchanged – it is the case of the gauge transformations.

The symmetry properties of the law sheaves are very powerful tools in the study of the physical laws, and also provide a measure of the mathematical beauty of a theory.

Definition 2.5. For a locally homogeneous law sheaf Λ over a spacetime $\mathcal{S}$, we can construct a group, from the germs $\Lambda(p)$ at a point p, by taking only the invertible germs of the law sheaf. This is the *local symmetry group* of the law sheaf. If Λ is only locally semi-homogeneous, then for each orbit of the pseudogroup of transformations of $\mathcal{S}$ we have a different local symmetry group.

In general, the law sheaf can be expressed as a selection of more levels of laws. The local symmetry group of each selector is different, but their intersection is the local symmetry group of Λ.

2.6. **The time**

Definition 2.6. A *time* is a topological space $\mathcal{T}$ with a totally order relation on it, compatible with the topology. A continuous map $\tau : \mathcal{S} \to \mathcal{T}$ induces a functor (we also denote it by τ) between the categories $\mathcal{O}(\mathcal{S})$ and $\mathcal{O}(\mathcal{T})$, named the *time coordinate* of $\mathcal{S}$.

As in the Remark 2.2, the time is allowed to be discrete, provided that its topology is discrete.

Definition 2.7. For any $t_0 \in \mathcal{T}$ we define the topological subspace $\tau^{-1}(t_0)$, and we name it the *space* at the instant t_0, corresponding to the time coordinate τ.

Principle 5. There exists (at least) a time coordinate $\tau : \mathcal{S} \to \mathcal{T}$.

It is possible to have more distinct time coordinates. For example, in the Newtonian Mechanics, a Galilei transformation can be given by the time translation, thus obtaining a reparametrization of the time coordinate. Moreover, in Special Relativity, there exist time coordinates that cannot be obtained one from another by time translation, because they correspond to distinct directions in the spacetime. In the cases of Special and General Relativity, the possible time coordinates are restricted by a causal structure. We will detail later the notion of causal structure.

2.7. **Worlds**

Let's summarize the principles enumerated so far:

The matter field μ is a global section in a matter fiber bundle $\mathcal{M} \stackrel{\sigma}{\to} \mathcal{S}$ over a spacetime topological manifold $\mathcal{S}$. There is a locally homogeneous subsheaf Λ of the sheaf of sections of the matter bundle $\Gamma(\mathcal{S}, \mathcal{M})$, named the law sheaf, containing the admissible matter fields. The matter field μ should be a global section of this law sheaf as well. We can identify it with the subsheaf it generates, $\mathcal{F}(\mu)$. All these principles are resumed in the following diagram:

$$\begin{tikzcd}
\text{Matter field} \arrow[r, hook, "\text{global section}", "\text{subsheaf}"'] & \text{Law sheaf} \arrow[r, hook, "\text{homogeneous}", "\text{subsheaf}"'] & \text{Matter sheaf} \\
& \text{Spacetime} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

or, using the notations we introduced,

$$\begin{tikzcd}
\mathcal{F}(\mu) \arrow[r, hook, "\text{global section}", "\text{subsheaf}"'] & \Lambda \arrow[r, hook, "\text{homogeneous}", "\text{subsheaf}"'] & \Gamma(\mathcal{S}, \mathcal{M}) \\
& \mathcal{O}(\mathcal{S}) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where $\mathcal{O}(\mathcal{S})$ is the category having as objects the open subset of $\mathcal{S}$, $Ob(\mathcal{O}(\mathcal{S})) = \{U \subset \mathcal{S} | U \text{ is an open set}\}$, and as arrows the inclusion maps, $Hom(\mathcal{O}(\mathcal{S})) = \{i : U \to$

$V|\, i(x) = x$ for all pairs $U, V \in Ob(\mathcal{O}(\mathcal{S}))$, $U \subset V\}$. The category $\mathcal{O}(\mathcal{S})$ defines the topological structure of $\mathcal{S}$.

Definition 2.8. A *world* consists in a topological space[2] $\mathcal{S}$ named the *spacetime*, a totally semi-homogeneous sheaf $\mathcal{M}$ on $\mathcal{S}$ (the *matter sheaf*), a locally semi-homogeneous subsheaf Λ of $\mathcal{M}$ named the *law sheaf* and a global section $\mu \in \Lambda(\mathcal{S})$ (the matter field).

Definition 2.9. A *locally homogeneous world* is a world having the spacetime $\mathcal{S}$ locally homogeneous.

If the spacetime is a topological manifold, then it is locally homogeneous. On the spacetime we can consider to have defined a differential structure. The differential structure is unique for dimension less than 4, but for dimensions higher than 3 it will no longer be unique.

Definition 2.10. We call a world with the spacetime being a differentiable manifold, and the matter sheaf being a sheaf of sections of a differentiable bundle over the spacetime, a *differentiable world*.

2.8. **Generalized worlds**

So far, the spacetime $\mathcal{S}$ has been taken to be a topological manifold, the matter bundle $\mathcal{M} \stackrel{\sigma}{\to} \mathcal{S}$ was a bundle over $\mathcal{S}$, the law sheaf Λ was a subsheaf of $\Gamma(\mathcal{S}, \mathcal{M})$, and the matter field μ a global section of Λ.

Theories of Physics, like the Classical Mechanics, Electromagnetism, Special Relativity, General Relativity, Quantum Mechanics, Dirac's Relativistic Quantum Mechanics, Gauge Theory, Quantum Field Theory etc. can be viewed as special cases of the model described above. Now we will make the natural generalization of the world defined before. We will see that some of the structures were defined having in mind this generalization.

A presheaf on the spacetime $\mathcal{S}$ is a covariant functor from the category $\mathcal{O}(\mathcal{S})^{op}$ to a category, for example the category *Set* of all sets as objects and functions as arrows.

The law sheaf is such a functor $\Lambda : \mathcal{O}(\mathcal{S})^{op} \to Set$, having an additional property that makes it to be a sheaf: namely that $(r_{U_i})_{i\in I}$ (consisting of the restriction maps) is an equalizer for the diagram

$$\Lambda(U) \overset{(r_{U_i})_{i\in I}}{\dashrightarrow} \prod_{i\in I} \Lambda(U_i) \underset{(f_i)_{i\in I} \mapsto (f_j|_{U_i\cap U_j})_{i,j\in I}}{\overset{(f_i)_{i\in I} \mapsto (f_i|_{U_i\cap U_j})_{i,j\in I}}{\rightrightarrows}} \prod_{i,j\in I} \Lambda(U_i \cap U_j)$$

for any open covering $U = \bigcup_{i\in I} U_i$ of an open set $U \subset \mathcal{S}$.

We can generalize the spacetime by employing instead of $\mathcal{O}(\mathcal{S})$ a more general category, such as a locale (Definition C.2). The law sheaf will be a homogeneous sheaf on S.

Definition 2.11. A *world* consists in a locale S (named *spacetime*), a concrete category M (named *matter*), a locally semi-homogeneous sheaf $\Lambda : S^{op} \to M$, and a global section μ of Λ.

[2] Or, more general, a locale (see Definition C.2).

We may reduce even more the conditions and extend the definition to Grothendieck sites, but for our purposes this is enough, for the moment.

2.9. **Examples of worlds**

The first examples are of locally homogeneous worlds.

Example 2.12. If we take the space as being $\mathbb{R}^3$ and the time $\mathbb{R}$, the spacetime will be $\mathbb{R}^{1+3}$. On this space, let's consider a vector bundle E with fiber $\mathbb{R}^k$, where $k \geq 1$. Let's consider the *wave equation*

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} = 0.$$

Its solutions form a homogeneous subsheaf of the sheaf $\Gamma(E \to \mathbb{R}^{1+3})$.

Example 2.13. Let the space be $\mathbb{R}^3$, and the line bundle E with fiber $\mathbb{R}$. The sheaf of the solutions of the *Laplace equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is homogeneous and it is a subsheaf of the sheaf $\Gamma(E \to \mathbb{R}^3)$. Note that this equation is time independent.

Similarly, we can consider the heat equation, the Maxwell equations and so on. If we consider complex bundles, we can define the Schrödinger equation, the Dirac and Klein-Gordon equations, the Yang-Mills equations etc. All these equations can be expressed by law sheaves, and therefore they describe worlds.

Example 2.14. In the General Relativity, the spacetime is a Lorentz manifold. The matter field can be considered to have two components, the metric, therefore the curvature, and the stress-energy tensor. The law sheaf will be the sheaf of the pairs (stress-energy tensor, metric tensor) respecting the Einstein equation. Of course, in General Relativity the spacetime topology itself is so much tied with the matter field, that it would be inaccurate to say that the matter field is secondary and the spacetime as a topological (and differentiable) manifold is primary. It is more like they are constraining each other.

Example 2.15. According to Quantum Mechanics, the state of a quantum system can be represented as a complex vector field – the wave function – which is a vector in a state space. The time evolution of a wave function is a solution of the Schrödinger equation. Since each measurement finds the state as being an eigenstate of the observable to be measured, two consecutive measurements impose initial conditions to the solution, that are in general incompatible. It is usually considered that this forces a discontinuity to occur between to consecutive incompatible measurements. In [39], I present a direct interpretation of Quantum Mechanics, that states that the wave functions, being piecewise solutions of the Schrödinger equation, are physical, and represents completely the quantum system. We can see that the spacetime is $\mathbb{R}^{3+1}$, and the law sheaf is given by the (local) piecewise solutions of the Schrödinger equation.

Example 2.16. In [40], I show that we can construct a Quantum Mechanics without recurring to discontinuities. In the Smooth Quantum Mechanics, the law sheaf is given by the (smooth) solutions of the Schrödinger equation, or of the Liouville - von Neumann equation, for the situations where the state is undetermined or entangled with another system. What appear to be wave function collapse is, in fact, a "delayed initial condition" imposed to the solution of the Schrödinger or Liouville - von Neumann equation. In that article, I show that, although the previous measurement fixed the initial condition of the observed system, the measurement device remains entangled with the system, until a new observation is performed. Thus, the second measurement determines, in fact, partially, delayed initial conditions for the entangled system composed by the previous measurement device and the quantum system to be observed. This scenario describes a world in which the matter field may never be determined, allowing each new measurement to add new initial conditions to a subsystem or another. There is a set of possible solutions of the evolution equation, and each measurement adds a new condition, reducing this set of solutions. It is possible even that all the measurements, past, present or future, never determine all the initial conditions. Therefore, the solution of the evolution equation (the matter section) may not be unique. The evolution is deterministic, because the wave function collapse takes place smoothly, only the initial conditions are not determined. Although it is a deterministic theory, it is compatible with free-will at the same extent as the standard Quantum Mechanics.

Remark 2.17. In the example 2.16, we need an enlarged definition of the world, in which the matter section is replaced by a subsheaf of the law sheaf. According to Remark B.2, any section in a sheaf of sets can be identified with a special type of subsheaf. We can therefore replace in general the matter section in the definition of a world by a subsheaf of the law sheaf, and recover the usual definition as a special case.

All the examples so far were only of the type PDE on vector bundles.

Example 2.18. Another class of examples (also of the PDE type) can be defined if we take the base manifold as being only composed by the time, and the fiber as being a *phase space*. We require the sections to respect the *Hamilton's equations*:

$$\left\{ \begin{array}{ll} \dot{p} = - & \frac{\partial \mathcal{H}}{\partial q} \\ \dot{q} = & \frac{\partial \mathcal{H}}{\partial p}, \end{array} \right. \tag{6}$$

where q and p are the generalized coordinates and momenta, and $\mathcal{H}$ the Hamiltonian. The variational principles expressed by the Hamilton's equations can be used to formulate the classical mechanics, as well as to describe electrodynamic and relativistic systems, particle interactions, quantum phenomena etc.

The previous example showed us that we can express the trajectories of particles using the World Theory, if we let the spacetime to be the time only, and move the positions in the fiber. This means that we move the space itself in the matter bundle. But can we keep the spacetime as the base manifold? One idea is to construct sheaves of trajectories. Would it be possible to construct a sheaf that associates to each open set of the spacetime a set of curves, so that they describe point-like particles? The sections of the sheaf can be characteristic functions of the sets representing the curves in the spacetime. But we required the law sheaf to be continuous, when we wanted it to be a subsheaf of a sheaf of continuous sections. We will show how we can do this in the next example, but first we need to recall the definition of the Sierpińsky space.

Definition 2.19. The *Sierpińsky space* $\mathbb{S}$ is a topological space formed by two points, $\{0,1\}$, with the topology $\tau = \{\emptyset, \{1\}, \{0,1\}\}$. The set $\{0\}$ is closed (but not open), and the set $\{1\}$ is open and not closed (its closure is $\overline{\{1\}} = \{0,1\}$).

Example 2.20. We can take the spacetime to be $\mathbb{R}^{1+3}$, and the fiber to be the Sierpińsky space $\mathbb{S}$. The continuous functions $f : \mathbb{R}^{1+3} \to \mathbb{S}$ are sections of the trivial bundle $\mathbb{S} \times \mathbb{R}^{1+3} \to \mathbb{R}^{1+3}$ (which we call the *Sierpińsky bundle*). $f^{-1}(1)$ is open in $\mathbb{R}^{1+3}$, and $f^{-1}(0)$ is closed. We can see that the continuous sections of the fiber bundle $\mathbb{S} \times \mathbb{R}^{1+3}$ can be identified in a natural way with characteristic functions of open sets of $\mathbb{R}^{1+3}$. We prefer here to identify the continuous sections f with the closed sets of the form $f^{-1}(0)$. Any continuous curve $\gamma : [0,1] \to \mathbb{R}^{1+3}$ is a closed set, therefore it defines sections of this bundle, as restrictions of the characteristic function of its complement. Therefore, the set of all possible trajectories of a particle can be represented by a subsheaf. If we consider a number n of particles, we simply consider n trajectories. The particle interactions such as disintegration and scattering can be expressed as unions of such curves, and they also form a subsheaf of the Sierpińsky sheaf. Adding constrains such as differentiability of the trajectories, and equations describing the variation of the position with the time, reduces this law sheaf. This example shows that not only the fields, but also the trajectories can be described as sheaves over the spacetime.

Example 2.21. We start with a Sierpińsky bundle $\mathcal{M} \to \mathcal{S}$ on a differentiable manifold $\mathcal{S}$ of dimension n (which can be 10 or 26 for example). The *strings* are 2-dimensional submanifolds of $\mathcal{S}$ which can be obtained as $f^{-1}(0)$ for some section f of $\mathcal{M}$, and they form a sheaf Λ_2. The *string theory* studies a subsheaf Λ of Λ_2 defined by action principles such as the Nambu-Goto action or the Polyakov action. The purpose of String Theory is to obtain a Quantum Theory of Gravity and to provide a description in terms of strings of the General Relativity and of the Standard Model of elementary particles.

Let us recall the definition of a dynamical system.

Definition 2.22. A *dynamical system* is a partial action of a monoid $(T,+)$ on a set M, (T, M, α):

(1) $\alpha : U \subseteq T \times M \to M$,
(2) $\alpha(0,x) = x$, and
(3) $\alpha(t_1, \alpha(t_2, x)) = \alpha(t_1 + t_2, x)$ for $(t_1, x), (t_2, x), (t_1 + t_2, x) \in U$,

where α is named the *evolution function* and M the *phase space* or *state space*. Particular cases of dynamical systems are measurable dynamical systems, topological dynamical systems, cellular automata, differential dynamical systems, which contains as a special case the Hamiltonian dynamical systems, etc.

Example 2.23. To a dynamical system (T, M, α) we can associate a world, such that the spacetime reduces to T, and the matter bundle is $T \times M$. If the dynamical system is real, then $T = (t_i, t_f) \subseteq \mathbb{R}$, and the topology is taken to be the induced one, if it is discrete (then $T = \mathbb{Z}$) or we don't have informations about its topology, we can consider the discrete topology. The law sheaf consists in the restrictions of the partial functions $\alpha_x : I(x) \to T \times M$, $\alpha_x(t) = (t, \alpha(t,x))$, where $I(x) = \{t \in T | (t,x) \in U\}$.

Using the Sierpińsky bundle we can define graphs on manifolds. Also, we can assign values to the vertices or edges of the graph, by considering a bundle with fibers $\mathbb{S} \times \mathbb{K}$, where $\mathbb{S}$ is the Sierpińsky space, and $\mathbb{K}$ is a field. We can also define triangulations into simplices, and associate to the manifold simplicial complexes. To the k-simplices we can associate values in the same manner.

If we want to obtain background independent description of the graphs and simplicial complexes, we can take as spacetime the *abstract simplicial complex* itself. The simplicial complex has a natural topology. We can therefore eliminate the need of the Sierpińsky bundle, and define a law sheaf on the new spacetime, obtaining a *background independent* theory. In most cases, the spacetime will no longer be locally homogeneous, but we still can take the law sheaf as locally semi-homogeneous.

Example 2.24. The *Regge calculus* is a way of replacing the Lorentz manifold in General Relativity with a simplicial complex. The metric is replaced by numbers associated to the edges (their lengths), and the curvature with angles of the 2-simplexes. Tullio Regge showed how we can translate in this setup the Einstein equation as constraints on these angles. His work led to significant applications in numerical relativity and Quantum Gravity.

Example 2.25. The *spin networks*, initiated by Roger Penrose [22, 23, 24, 25], are graphs having all vertices of order 3, and the edges labeled with integers satisfying a set of rules (the *triangle inequality* and the *fermion conservation*). Later, the spin networks were generalized, by replacing the numbers on the edges with group representations, the vertices with intertwining operators, and by allowing the order of each vertex to be greater than 3 [34, 2, 3]. All these constructions can be reformulated in the context of the World Theory.

Example 2.26. By employing instead of a continuous spacetime, a lattice of vertices, together with edges and faces, and by defining the fields only at vertices, and elements of a Lie group on edges and loops, we can construct a *lattice gauge theory*.

Example 2.27. Starting from the observation that in General Relativity the causal structure contains, up to a conformal transformation, all the information, Sorkin [4, 35, 36, 37, 27] initiated the idea of *causal sets*. In this theory, we keep only a discrete set of points of the Lorentz manifold of the General Relativity, and a partial order relation encoding the causal structure. A causal set have an order relation which is irreflexive, transitive, and between any two points there is a finite number of intermediate points.

These examples showed us that, in general, the theories in Physics respects a pattern – they can be expressed in terms of the mathematical structure named *world*. This works for continuous, differentiable, as well as discrete models of spacetime.

3. **Causal worlds**

3.1. **Causal structures**

We considered so far the spacetime $\mathcal{S}$ as containing both the space and time, without differentiating them too much. In the nonrelativistic theories, such as the Newtonian Mechanics, the spacetime is considered as the product manifold $\mathbb{R}^3 \times \mathbb{R} = \mathit{Space} \times \mathit{Time}$.

If the space is considered as absolute, then the decomposition *Space* $\times$ *Time* is unique. If, by contrary, we cannot distinguish the move of an observer relatively to the space, then any other decomposition *Space* $\times$ *Time'*, will be as good as the first one. This is the case in Newtonian Mechanics. In relativistic theories, the decomposition *spacetime* = *Space* $\times$ *Time* will also not be unique, but even more, we can have decomposition with different space: *Space'* $\times$ *Time'*. In the non-relativistic theories, it is enough to compare the time coordinate of two events to see if one of them is in the past of the other, and consequently if it can "influence" it. In the relativistic theories, two spacetime events are separated by a space-like, time-like or light-like interval. The *past* of an event is constituted by all the events from which this one is separated by a non-space-like interval and have the time coordinate smaller (in any coordinate map of a time-oriented atlas). The *Principle of Causality* states that the value of the matter field in each event should depend only on the matter field in events that are in the *past* of the considered event.

One can express the causal structure by a relation of preorder on the set of the spacetime events – the relation b *is in the past of* a, denoted by $b \to a$.

Definition 3.1. A *causal relation* $\to$ on a spacetime $\mathcal{S}$ is a relation of preorder $\to$ on $\mathcal{S}$, such that, for any event $a \in \mathcal{S}$, the set $Past(a) := \{b \in \mathcal{S} | b \leq a\}$ is closed. The set $Past(a)$ is named the *causal past* of a.

Definition 3.2. To events a and $b \in \mathcal{S}$ are said to be *simultaneous* if $a \to b$ and $b \to a$.

Remark 3.3. In the Theory of Relativity, there are no simultaneous events, while in the Newtonian Mechanics, two events are simultaneous if and only if their times are equal.

Remark 3.4. We can use the relation $Past(b) \subseteq Past(a)$ instead of the relation $b \to a$. Because if one point b is in the past of another one a, then all the points in the past of b will also be in the past of the point a, we see that all the sets of the form $a_+ := \mathcal{S} - Past(a)$ for some $a \in \mathcal{S}$ form a subcategory (which is full) of $\mathcal{O}(\mathcal{S})$. This suggest the following definition of a causal structure.

Definition 3.5. Let $\{\mathcal{C}_i\}_{i \in I}$ be an open covering of the spacetime. For a spacetime event a we define the set

$$a_- := \bigcap_{a \notin \mathcal{C}_i,\, i \in I} (\mathcal{S} - \mathcal{C}_i). \tag{7}$$

If for each $\mathcal{C}_i$ there exists a spacetime event a such that $\mathcal{C}_i = a_-$, then the subcategory $\mathcal{C} \subset \mathcal{O}(\mathcal{S})$ generated by $\{\mathcal{C}_i\}_{i \in I}$ is named a *causal structure* for the spacetime.

Propositon 3.6. There is a one-to-one correspondence between the causal relations and the causal structures on a spacetime.

Proof. If $\mathcal{C}$ is a causal structure on the spacetime, then for any two points a and $b \in \mathcal{S}$ one can define the relation $b \to a$ iff $b \in a_-$. The set of all $\mathcal{C}_i$ not containing b contains the set of all $\mathcal{C}_i$ not containing a, because $b \in a_-$. Therefore, $b_- \subseteq a_-$, and this means that every point in the past of b is also in the past of a.

Conversely, we construct the open sets C_i as the sets $\mathcal{S} - Past(a)$ for all a in spacetime, and they satisfy the definition of a causal structure. $\square$

The topology generated by the past sets will be weaker than the topology of the spacetime. This implies that there may be points that are not separated causally – corresponding to the simultaneous points.

Remark 3.7. We have given the definition with open sets above for the causal structure not only because it contains implicitly the continuity and the partial order causal relation. The main reason is that it expresses the causal structure as a subcategory of $\mathcal{O}(\mathcal{S})$, and this will work for more general definitions of a world, on more general spacetimes than topological spaces.

Remark 3.8. So far, the causal structure was introduced as an extra structure on the world. We did this because we wanted to introduce the ideas gradually. In some important situations the causal structure is suggested by the very structure of the world, and in some cases even emerges in a natural way (for example in the General Relativistic cosmological models).

The time has been defined as a totally ordered topological space $\mathcal{T}$, with the order being compatible with the topology.

Definition 3.9. A time structure τ is *compatible* with the causal structure $\mathcal{C}$ if, for any two events $a, b \in \mathcal{S}$, if $b \to a$ then $\tau(b) \leq \tau(a)$.

Remark 3.10. It is possible to have more than one distinct time structure compatible with the same causal structure. For example in Special Relativity, any frame will contain a temporal vector, and the coordinates defined by that vector defined indeed a time structure.

Remark 3.11. In General Relativity, it is possible to have closed timelike curves, and the causal structure can no longer be considered obtainable from a causal preorder relation. Yet, we can use a weaker concept of causality. For such situations, we can use instead a *local causal relation*, as being defined as a relation for which there exist an open covering of $\mathcal{S}$, such that its restrictions to the open sets in the covering are causal relations. We define accordingly the *local causal structure.*

3.2. Causal worlds

Definition 3.12. A *causal world* is a world with a causal structure on its spacetime.

Because we have defined the causal structure and the time in terms of categories, and because the time coordinate is a functor, their definitions extends straightforward to the more general definition of a world.

We may wonder if we shouldn't add more conditions to the causal structure. It may seem that the causal structure must do something more than simply be compatible with the topology of the spacetime. The following example presents a situation in which the law sheaf itself determines almost uniquely a canonical causal structure.

Example 3.13. Let's consider the world as being a time orientable Lorentz manifold M of dimension $n+1$, $n \geq 2$. Then there exist two canonical causal structures, corresponding to each of the possible time orientations. If the dimension is $1+1$, the metric $-g$ provides also a Lorentz structure, and we have four distinct canonical causal structures. An energy condition can rule out two of them, by forbidding the tachyons. In both cases $n \geq 2$ and $n = 1$, if the matter fields on the Lorentz manifolds are subject to a large entropy disequilibrium, such as The Second Law of Thermodynamics, then it is possible to select the time direction in which the entropy increases. In this case the causal structure will be unique, as it seems to be in our world.

In the example above, the metric is a component of the matter field. The matter field in the sense of the World Theory contains the metric, but also the stress-energy tensor. If we want to include a Maxwell field (to obtain an Einstein-Maxwell space), Dirac fields or Yang-Mills fields in the matter field, we can do so, but the causal structure is determined by the metric, which is only a part (a component) of the full matter field. The metric determines on the tangent spaces the light cone, which defines a causal relation between the points of the tangent bundle. This relation, in turn, defines geodesics which define a causal relation locally, between the points of the manifold. Additional conditions of time orientability and positive energy may be required, and we remain with only two causal structures. If the entropy increases towards a time direction, we consider that direction to select only one causal structure.

The General Relativistic cosmological models seems to respect such a condition. The metric is itself (part of) the matter field, and it restrains the number of interesting causal structures.

These examples suggest the following problem about the causality: Is there a general and natural way in which the matter field of a world constrains or even determines the causal structure?

3.3. **The principle of causality**

Roughly speaking, a Principle of Causality will state that the value of a matter field at a point a of spacetime depends on its values in events in the past of a, and not on the values in events in the future, or which are in no causal relation with a. This seems clear and obvious, but it may not be so at a second thought. Let's consider two events $b \to a$ in spacetime. The value of the matter field in a, $\mu(a)$, will depend on its value in b, $\mu(b)$. But in fact this can be interpreted also backwards: the value of the field in b depends on its future value $\mu(a)$. One can see that in fact it is not a unilateral dependence, but rather a bilateral relation (a correlation). The causal structure provides this bilateral relation with an orientation, and this is why the correlation appears to be unilateral. Logically, if we replace the causal structure with it's dual - where all the causal relations are reversed, what we obtain is a causally dual world where the Principle of Causality will said that $\mu(b)$ depends on $\mu(a)$. Therefore, whenever we see that the value of the matter field at an event b depends on a future value $\mu(a)$, we can just reinterpret and say that the future value $\mu(a)$ is the one which depends on the past value $\mu(b)$.

Definition 3.14. Let us consider a world whose spacetime $\mathcal{S}$ is compatible with a causal structure $\mathcal{C}$. We say that two events a and b of $\mathcal{S}$ are *causally related* or *causally dependent* if $a \to b$ or $b \to a$.

The principle of Causality can be expressed into the sheaf language like this:

Principle of Causality. Let's consider a world whose spacetime $\mathcal{S}$ is compatible with a causal structure $\mathcal{C}$. For any two topologically separated events $a, b \in \mathcal{S}$ not related causally, and any germs $\rho_a \in \Lambda(a)$ and $\rho_b \in \Lambda(b)$ of the law sheaf, there exist a global section $\rho \in \Lambda(\mathcal{S})$ of the law sheaf, such that the germs of ρ in a and b are equal to ρ_a and ρ_b.

4. Determinism and indeterminism

4.1. Deterministic and indeterministic worlds

We define now a deterministic world as a world in which any event is determined by the events in its own past.

Definition 4.1. A *deterministic world* is a causal world with the property that any global section $\rho \in \Lambda(\mathcal{S})$ is uniquely determined at any event A in the spacetime by its restriction to the set $Past(A)$.

Definition 4.2. A *strongly deterministic world* is a causal world such that for any time coordinate $\mathcal{S} \xrightarrow{\tau} \mathcal{T}$ on the spacetime $\mathcal{S}$, for any $t_0 \in \mathcal{T}$, any global section $\rho \in \Lambda(\mathcal{S})$ is uniquely determined by its germs on $\tau^{-1}(t_0) \cap Past(A)$.

The worlds based on classical PDE's such, as the wave, heat, and Laplace equations, are deterministic, for a suitable causal structure, because the solutions are determined by the initial or boundary conditions. The Newtonian Mechanics is the prototype of a strongly deterministic world. The Special Relativity, when deals with classical fields (fluids, point particles, electromagnetic fields) is deterministic. In Quantum Mechanics, the evolution of a particle is described by the Schrödinger, Klein-Gordon and Dirac equations, which are strongly deterministic. The state vector reduction, which takes place when the system is measured, is, at least in appearance, nondeterministic. Examples of deterministic Quantum Mechanics are the Bohmian mechanics [5, 6], and the Smooth Quantum Mechanics [40]. Some versions of the Many World interpretation [16, 17, 12, 13, 10, 11] are deterministic at the level of the Multiverse, where the collapse is considered not to occur, but each independent world is still indeterministic.

4.2. Determinism and prediction

The physicists' aim is to describe the world and its laws. They create theories that explain the phenomena and make predictions about the outcomes of experiments. The theories cannot be proved, but they can be rejected. The predictions are the means to verify a scientific theory (as Laplace told to Napoleon when he offered his *Traité de Mécanique Céleste*, or as Popper wrote in [26]). A theory is considered better when it makes stricter predictions, because this increases its falsifiability. It is easy to see that when a theory predicts the values for all the parameters of the system under consideration, the prediction is better and the falsifiability increases. This implies that when a theory is deterministic, it is more falsifiable, more complete, and therefore better scientifically, than when it is nondeterministic. On the other hand, until the rise of the Quantum Mechanics, all the theories in Physics were deterministic, or explainable in terms of deterministic ones. This is why Laplace was so delighted about the power of prediction of the Classical Mechanics.

4.3. **Relativity and determinism**

In general it is believed that the Theory of Relativity is deterministic. Both Special and General Relativity are compatible with nondeterministic fields. For example, the Dirac equation is deterministic, but the state vector reduction is not (at least in the Copenhagen interpretation).

Arnowitt, Deser and Misner (ADM) formulated in [1] the General Relativity as an initial value problem, with the initial data on a spacelike hypersurface Σ. The Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi k T_{\mu\nu}, \tag{8}$$

splits in an evolution equation having 6 components, and a 4-components constraint for the metric on Σ. The evolution equation is deterministic, as long as the stress-energy tensor, which determines the curvature of the spacetime, is associated to deterministic fields.

On the other hand, the stress-energy tensor $T_{\mu\nu}$ can be regarded as being determined by the curvature of the Lorentz spacetime M. Because we can modify the metric and even the topology of the manifold, in some open set in the future of a spacelike hypersurface Σ, we can have an infinity of solutions with the same initial data on Σ. The ADM formalism implies only that when the stress-energy is associated to a deterministic field, the evolution of the manifold itself is deterministic, at least in a local fashion[3].

We can conclude that the Relativity is compatible[4] with both the determinism, and with the indeterminism, depending on how the matter fields are. When only classical fields are considered, the Relativity Theory is, nevertheless, deterministic, and perhaps this made Einstein a strong supporter of the determinism.

5. **Experiment, theory, reality**

5.1. **From experiment to theory**

Let's suppose we want to study a system, or a phenomenon. When doing this, we measure the values of some quantities. We collect experimental data. We observe the objects composing the system. Later, we will try to describe them by a theory, or at least by a law.

If we consider the system under research as being a world, we have to collect three kinds of data. The first is the base space (the spacetime). Then, the matter sheaf, and the third is the matter section. They are interdependent, and we have to alternate the theoretical descriptions with the experiments, because the theory allows us to select what data to collect, and the new data allow us to confirm or reject, or to adjust the theory.

[3] For nontrivial topologies, the things get more complicated.

[4] Another example, using causal sets, is provided by Rafael Sorkin[38].

The elaboration of the theory, or of the law describing the phenomena under research, depends on the experimental data. We will try to find the simplest possible theory. The simplicity of the theoretical description is given by the length of the description in a language, and it depends on the language. The *Kolmogorov-Chaitin (descriptive) complexity* [9, 19] is a way to express the complexity of the theoretical description. In fact, it measures the minimal length of a programming language that can be used to generate a string containing the description. It depends on the programming language, and it is in a way paradoxical, because given two strings s_1 and s_2, it is possible that s_1 is simpler than s_2 in a language L_1 and s_2 is simpler than s_1 in another language L_2. This can mean that the complexity is somehow "subjective", depending on the language used to express it.

A more practical way to measure the complexity of a description was elaborated in 1978 by Jorma Rissanen [28, 29, 30, 31, 32, 33]: the *minimum description length* principle (MDL). The MDL principle allows a stronger language independence of the definition of complexity.

The idea of complexity can be applied to understand the role of a theory to systematize and corroborate the experimental data. If the experimental data respects one rule, even if only partially, we can use that regularity to compress the description of that data. The rules applied in the compression provide us a theoretical description. A better theory is simpler, this meaning that it has a smaller description length.

The simplest set of rules, or description, or theory, can be infirmed by further experimental data. We don't have the guarantee that we have found the real law. We can't even know whether there exists such a law. At least we have found a good mnemonic. We can describe the observations as resumed by a law, and this helps us to think abstractly about the data, and also to explain and communicate ideas to other persons. In fact, the initial application of the MDL principle was in the learning theory. In the case of the theories, we can say that scientists try to learn the rules of the phenomena, at least the apparent rules.

Let's suppose we know the structure of the matter bundle (the base spacetime, the fiber, the structural group). When we measure the value of the matter field at a given point, the experimental error affects the knowledge of the position and time, and of the value of the matter field. But we can say that, even by knowing them within an error, we have got a condition, which restricts the sheaf of admissible sections. All the experimental data we collect adds new conditions, and hence defines a subsheaf within the matter sheaf – an *experimental sheaf.* Of course, this is made under the assumption that we know the structure of the matter bundle, which is in fact itself subject of the research.

The sheaf approach also allows us to go from quantitative to qualitative, because the sheaf associates to each open set of the spacetime an object in a category. An object in a category contains more structured information than a point in a vector space. This, on the one hand, provides more generality, by allowing the qualitative study of the phenomena. On the other hand, it complicates the solutions, adding the necessity of determining the category of objects associated to the spacetime open sets.

Assuming that we can determine experimentally the matter field μ (as well as the base space and the matter sheaf), we want to find the most appropriate law sheaf Λ admitting μ as global section. This problem is fundamental for Physics. In the same time, it is widely accepted that we cannot know for sure if the law sheaf is the "true" law of Nature. This is clear when we don't know the matter field at all points. We can only know its values in a limited (most likely finite) number of points of the spacetime. At each point we only know approximations of projections of the matter field (only for some of its "components"). And our knowledge is limited in time, simply because we can only know the near past, and the distant future values are still open. And our knowledge is limited in space too. But even if we would know exactly and completely the matter field, we still need some informations about the matter bundle, or we need to extract/propose the physical laws. And here too will be many possible answers, and we will prefer the one providing the simplest and better explanation.

There will be an infinity of locally homogeneous sheaves admitting μ as section. We will keep the smaller one, the simpler one, or the best fitted for our needs.

5.2. **Unification of Physics**

The World Theory provides a common (and unified) framework for the theories, not a unification of Physics. But it can clarify several conditions that a unified theory may be required to fulfill.

Completeness. The first requirement is that the unified theory describes all the observed phenomena, at any physical level. All the matter fields and the laws governing their evolution can be considered as global sections of sheaves over the spacetime. If all the law sheaves needed in this description have the same base spacetime, they can be joined in a unique sheaf. The relations and the interactions among various matter fields can be also included.

Minimality. The picture above is not enough, the unified theory should also eliminate redundancy and express, as much as possible, constants in terms of other constants, and fields in terms of other fields, to reduce their number to a minimum.

Simplicity. This principle has both aesthetical and practical meanings. The simplicity makes the theory more beautiful, more comprehensible and trustable. It is expressible by the means of Kolmogorov-Chaitin complexity.

We also require that the unified theory is falsifiable ([26]), by making testable predictions.

The World Theory can be a good framework to describe characteristics we expect from a theory, and to compare different theories, to see what we can change at a theory and still keep what was successful.

5.3. **What the World Theory can do?**

5.3.1. **Model theories in Physics**

The mathematical object named world can be used to model mathematically the laws of physics. We have seen that a very large class of theories of physics can be expressed in this language. This is because it abstracts the most general properties of the known theories. The World Theory's main purpose is to derive logically general conclusions that can be applied to a large class of physical theories. This means that it is not opposed to the other theories, as a competitor, and does not makes predictions other that the other theories do. It is not likely to be falsifiable, but this is not a handicap, because this theory doesn't try to eliminate from competition other theories. It's purpose is only to extract the common assumptions in an abstract manner, and to derive logical and mathematical consequences.

5.3.2. **Model emergent phenomena**

The second purpose is to provide a way to mathematically formulate theories from other domains of Science, such as Chemistry, Biology, Psychology, Economics. The World Theory being a mathematical one, it can be applied to any system that respect its definition. The definition is very broad, and many fields will fit it. This raises the question whether it is too general to be useful. I think that whenever we will need to specialize it more, we can do it, and at least we have a basis. These theories study laws that cannot be directly inferred from the laws of Physics – they study *emergent phenomena*. Although it is supposed that we can describe these emergent objects and laws in terms of physical objects and laws, we may not need this, or we simply may prefer to emphasize what is "over" the matter fields of Physics. We can compare these emergent laws with a computer program, which is a computer program without regard of its physical support, or whether or not it runs on a hardware. It is made of binary digits, but it is something more than those digits. Thinking at it in terms of digits is not always helpful, although it is true that it is just a sequence of 0 and 1. In the same way, some phenomena can exist in a world of matter fields, but the emergent structures add new constrains on the law sheaf, and even endow it with extra structures. These structures can be described mathematically. In general, as mathematical structures, they can be described as objects in some categories. But the law sheaf is a functor, it can be valued in the new category now, therefore it gains an enriched structure. If initially it was, for example, a sheaf of sets, we may upgrade it to a sheaf of abelian groups; if it was a sheaf of abelian groups, it may be upgraded to a sheaf of modules, or we may construct equivalence classes and obtain other structures. The more complex objects are sections of more structured sheaves.

5.3.3. **Provide mathematical ground for various conceptions of the world**

A third purpose is to offer a less ambiguous way to discuss about concepts that usually fit in areas of Philosophy, such as Metaphysics and Epistemology, and this is another direction that can be developed in the future. A philosophically neutral, unitary and unambiguous language provided by Mathematics can bring more clarity in such fields.

Appendix A. Locally homogeneous sheaves

In this section I will introduce the locally homogeneous sheaves and presheaves, which can be used to generalize the type of constraints imposed by the PDE system. A *locally homogeneous presheaf* over a topological space X is a presheaf that is, in a way, similar in all points of X. The local homogeneity condition extracts one of the essences of the PDE systems, their independence of the position. In order to define the locally homogeneous presheaves and sheaves, we have to recall the notion of pseudogroup of transformations (developed by Élie Cartan ([7, 8])) and to introduce the locally homogeneous spaces. A presheaf is locally homogeneous if its pseudogroup of transformations acts transitively. A homogeneous space has the pseudogroup of transformations obtained by restrictions and unions of transformations from a group of transformations. The homogeneous presheaves and sheaves are natural objects of the Klein geometries, so we can consider the locally homogeneous presheaves and sheaves as the natural objects of a generalization of the Klein geometries. We will need these ideas in the appendix B, when we will study the sheaf selection, a generalization of the initial conditions for PDE, as a way to identify a global section of a locally homogeneous sheaf.

A.1. **Pseudogroups of transformations**

Definition A.1. Let X be a topological space. A *transformation* is a homeomorphism $f : U \to V$, for open sets $U, V \subseteq X$. $\mathrm{dom} f = U$ is named the *domain*, and $\mathrm{im} f = f(U) = V \subseteq X$ the *image* of the transformation f. A transformation being a relation, we can define the union and the intersection of an arbitrary number of transformations. An arbitrary set of transformations is said to be *compatible* if their union is again a transformation.

Definition A.2. A *pseudogroup of transformations* of X is a set $\mathcal{T}$ of transformations of X such that:

(1) For any nonempty open set $U \subseteq X$, the identity $1_U : U \to U$ is in $\mathcal{T}$.
(2) If $f : U \to V$ and $g : V' \to W$ are in $\mathcal{T}$ and $V \cap V' \neq \emptyset$, then the transformation $g \circ f : f^{-1}(V \cap V') \to g(V \cap V')$ is in $\mathcal{T}$.
(3) If $f : U \to V$ is in $\mathcal{T}$ then $f^{-1} : V \to U$ also is in $\mathcal{T}$.
(4) If $f : U \to V$ is a transformation and $U = \bigcup_{i \in I} U_i$ a covering, such that for all $i \in I$, $f_i := f|_{U_i} \in \mathcal{T}$, then $f \in \mathcal{T}$. In other words, the union of any compatible set of transformations from $\mathcal{T}$ is again in $\mathcal{T}$.

Remark A.3.

(1) As a consequence of the first condition, the identity $1_X \in \mathcal{T}$.
(2) The second condition yields that if the composition of two transformations from $\mathcal{T}$ is defined ($V \subseteq V'$), then it is in $\mathcal{T}$. In particular, using the first condition, it follows that for any open $\emptyset \neq U' \subseteq U$ and any transformation $f : U \to V$, the restriction $f|_{U'} : U' \to f(U')$ is a transformation of $\mathcal{T}$, because $f|_{U'} = f \circ 1_{U'}$.
(3) A pseudogroup of transformations $\mathcal{T}$ of a topological space X is identical with a presheaf of invertible continuous sections of the trivial topological bundle $X \times X$. In general the former is not a sheaf.

Definition A.4. A pseudogroup of transformations $\mathcal{T}$ of a topological space X is said to be *transitive* if for any $x, y \in X$ there exists $f \in \mathcal{T}$ such that $f(x) = y$.

Propositon A.5. The set of all local homeomorphisms $f : U \to V$ of a topological space X forms a pseudogroup of transformations $\mathcal{T}(X)$.

Proof. For any $\emptyset \neq U \subseteq X$, 1_U is a local homeomorphism. The inverse of any local homeomorphism is a local homeomorphism. When exists, the composition of two local homeomorphisms is a local homeomorphism too. By definition, the union of an arbitrary set of transformations is in $\mathcal{T}(X)$ if and only if the transformations are compatible. $\square$

Definition A.6. A topological space X is said to be a *locally homogeneous topological space* if its pseudogroup of transformations $\mathcal{T}(X)$ is transitive.

Example A.7. As an example of a locally homogeneous topological space we can take a topological manifold X, because for any $x, y \in X$ there is a transition map f (which is a locally homeomorphism) such that $f(x) = y$.

Definition A.8. A pseudogroup of transformations $\mathcal{T}$ of a topological space X is said to be *global* if for any local transformation $f \in \mathcal{T}$, $f : U \to V$, U can be expressed as a union of opens $U = \bigcup_{i\in I} U_i$ such that each $f_i := f|_{U_i}$ extends to a global homeomorphism $\tilde{f}_i : X \to X$. In this case, the global homeomorphisms from $\mathcal{T}$ form a *group of transformations* of X.

Definition A.9. Let $\mathcal{T}$ be a pseudogroup of transformations acting transitively on a locally homogeneous topological space X. The pair $(X, \mathcal{T})$ is named *locally homogeneous space.* If $\mathcal{T}$ is a global pseudogroup of transformations, then the pair $(X, \mathcal{T})$ is named *homogeneous space.*

A.2. Locally homogeneous presheaves

Definition A.10. Let $\mathcal{F}$ be a presheaf over a locally homogeneous topological space X. Let U be an open set, $f : U \to V$ a local homeomorphism. We say that f *preserves* the presheaf $\mathcal{F}$, or that $\mathcal{F}$ is an *invariant* of the local homeomorphism f, if the direct image or push forward presheaf $f_*\mathcal{F}(U)$ is isomorphic to the presheaf $\mathcal{F}(V)$. We denoted by $\mathcal{F}(U)$ the restriction of the presheaf $\mathcal{F}$ to U.

Remark A.11. $f : U \to V$ being homeomorphism, the pullback or inverse image $f^{-1}\mathcal{F}(V)$ is also isomorphic to $\mathcal{F}(U)$.

Definition A.12. Let $\mathcal{F}$ be a presheaf on a topological space X. The set of local homeomorphisms of X preserving $\mathcal{F}$ generates a pseudogroup of transformations denoted $\mathcal{T}(\mathcal{F})$.

Remark A.13. It is important to remember that $\mathcal{F}$ is not invariant to all the local transformations of $\mathcal{T}(\mathcal{F})$. $\mathcal{T}(\mathcal{F})$ contains the transformations which preserves $\mathcal{F}$, but also any transformation of X which is a union of transformations preserving $\mathcal{F}$. The union may not preserve $\mathcal{F}$. The following counterexamples provide such situations.

Example A.14.

(1) Let $X = X_T \cup X_M$ be the topological space having two connected components X_T and X_M, each one of them being homeomorphic to S^1 (see figure 1, left). Let's consider a line bundle $E \to X$, with fiber $\mathbb{R}$, such that its restriction to X_T, $E_T \to X_T$, is the trivial line bundle $X_T \times \mathbb{R}$, and its restriction to X_M, $E_M \to X_M$, is a Möbius line bundle. We take U_T, V_T connected open sets such that $X_T = U_T \cup V_T$. A homeomorphism $f : X_T \to X_M$ provides a similar covering $X_M = U_M \cup V_M$, where $U_M = f(U_T)$ and $V_M = f(V_T)$. The restrictions $f|_{U_T}$

and $f|_{V_T}$ are local transformations preserving the sheaf of sections of the line bundle, but f itself is a local transformation of X, $f = f|_{U_T} \cup f|_{U_M}$, which does not preserves the sheaf structure.

(2) We can use the previous counterexample to construct another one, based on a connected topological manifold (as in figure 1, right). Let $E_M \to X_M$ be a Möbius real line bundle as in the previous example. The manifold $X = X_M \times (0,1)$ admits a line bundle $E = \pi_1^*(E_M \to X_M)$, where $\pi_1 : X_M \times (0,1) \to X_M$ is the canonical projection. After fixing a point $x_0 \in X_M$, we can define a coordinate chart $h_0 : X_M - \{x_0\} \to (0, 2\pi)$, which induces a natural coordinate chart $h : (X_M - \{x_0\}) \times (0,1) \to (0,2\pi) \times (0,1)$, $h(x,y) := (h_0(x), y)$. The natural riemannian metric on $(0,2\pi) \times (0,1)$ induces via h a metric on X, and a distance d. Let $\epsilon_1 = 0.1$, $\epsilon_2 = 0.2$, $p := h^{-1}((\pi, 0.5))$, $U = X_M \times (\epsilon_1, \epsilon_2)$ and $V = \{x \in X | \epsilon_1 < d(p,x) < \epsilon_2\}$. Let's consider a homeomorphism $f : U \to V$. f can be expressed as a union of local transformations preserving the line bundle structure (therefore the sheaf structure), but f itself does not preserves it.

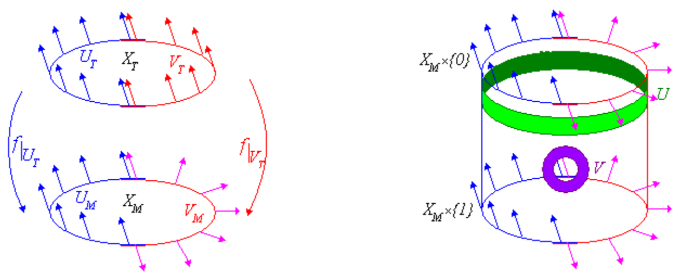

FIGURE 1. $\mathcal{F}$ is not invariant to all the local transformations of $\mathcal{T}(\mathcal{F})$.

Definition A.15. A presheaf $\mathcal{F}$ over the locally homogeneous topological space X is said to be *locally homogeneous* if and only if its pseudogroup of transformations $\mathcal{T}(\mathcal{F})$ is transitive. A presheaf which is not locally homogeneous is said to be *inhomogeneous*. A locally homogeneous presheaf $\mathcal{F}$ whose pseudogroup of transformations $\mathcal{T}(\mathcal{F})$ is global is said to be *homogeneous* (or globally homogeneous).

Remark A.16.

(1) If $\mathcal{F}$ is a (locally) homogeneous sheaf over the topological space X, then it defines on X a structure of (locally) homogeneous space $(X, \mathcal{T}(\mathcal{F}))$.
(2) If X is a homogeneous topological space, then the sheaf $\mathcal{T}(X)$ of its local transformations is locally homogeneous. The reciprocal statement follows from the definition A.15.
(3) Let $\mathcal{F}$ be a homogeneous sheaf on a topological space X. The pseudogroup $\mathcal{T}(\mathcal{F})$ is obtained by restricting global transformations from a group G of transformations of X. Moreover, G acts transitively on X (X is what is called a G-space). In other words, $\mathcal{F}$ is a presheaf invariant to the transformations of a G-space X.

Definition A.17. Let $\mathcal{T}$ be a pseudogroup of transformations of a topological space X. A presheaf $\mathcal{F}$ on X is said to be $\mathcal{T}$-invariant if $\mathcal{T}$ is a subpseudogroup of $\mathcal{T}(\mathcal{F})$.

Remark A.18. The definition A.17 saids that if any transformation from $\mathcal{T}$ can be expressed as a union of transformations preserving $\mathcal{F}$, then $\mathcal{F}$ is said to be $\mathcal{T}$-invariant.

Definition A.19. A presheaf $\mathcal{F}$ over a locally homogeneous topological space X is named *totally homogeneous* if it is $\mathcal{T}(X)$-invariant.

Remark A.20.

(1) The definition A.19 saids that the presheaf $\mathcal{F}$ is preserved by any local transformation of X.
(2) A totally homogeneous presheaf is locally homogeneous, but not necessarily globally homogeneous.
(3) The continuous sections of a topological fiber bundle form a totally homogeneous sheaf, because of the property of local triviality.

The locally homogeneous sheaves can be defined only on locally homogeneous topological spaces. It may be interesting to have the closest notion to local homogeneity of sheaves on spaces that are not locally homogeneous.

Definition A.21. A presheaf $\mathcal{F}$ over a topological space X is named *locally semi-homogeneous* if the orbits of its pseudogroup of transformations $\mathcal{T}(\mathcal{F})$ coincides with the ones of $\mathcal{T}(X)$.

Definition A.22. A presheaf $\mathcal{F}$ over a topological topological space X is named *totally semi-homogeneous* if it is $\mathcal{T}(X)$-invariant.

Remark A.23. A locally semi-homogeneous sheaf $\mathcal{F}$ over a locally homogeneous topological space X is itself locally homogeneous, because $\mathcal{T}(X)$ has exactly one orbit. A totally semi-homogeneous sheaf over a locally homogeneous spacetime is totally homogeneous.

A.3. Examples of locally homogeneous presheaves

Example A.24. If X, Y are two topological spaces, the continuous functions defined on opens of X to Y form a sheaf, the sheaf of continuous sections of the trivial bundle $X \times Y \xrightarrow{\pi_1} X$. If X is locally homogeneous, then this sheaf is locally homogeneous.

Example A.25. If M is a (real or complex) differentiable manifold, then the differentiable functions defined on opens of M and valued in the field $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ form a locally homogeneous sheaf. If M has boundary it is not locally homogeneous, but it is semi-homogeneous.

Example A.26. Let $E \to M$ be a vector bundle over a differentiable manifold M. The local sections of E form a locally homogeneous sheaf over M, because the conditions of local triviality from the definition of a vector bundle. The exterior bundle $\wedge E$, the dual vector bundle E^* of E, the bundles of tensor of type (r, s) over E, all these associated vector bundles satisfy the local triviality condition, therefore their sections form locally homogeneous sheaves.

Example A.27. Let $(E, g) \to M$ be a semiriemmanian vector bundle over a differentiable manifold M. The local frames on E form a sheaf over M. The g-orthogonal local frames form a sheaf too. Because of the local triviality, the sheaf of local frames and the one of local g-orthogonal frames are locally homogeneous. The $(1,1)$ tensor fields preserving g form a sheaf $O(E,g)$, which is also locally homogeneous.

The following is an example of sheaf that is not necessary locally homogeneous.

Example A.28. Let M be a semiriemannian manifold, and TM its tangent bundle. The g-orthogonal local frames form a sheaf. The previous example saids that the sheaf of $(1,1)$ tensor fields preserving g form a sheaf $O(TM,g)$ which is locally homogeneous. On the other hand, in this case, extra information is provided by the fact that TM is a tangent bundle, namely, the coordinate changes of the manifold M and the frame changes of TM are not independent, as in the case of a generic semiriemannian vector bundle. In the case of the tangent space there exist preferred local frames, the holonomic frames, which express the relation between the vector frames of TM and the local coordinates on M. The local homeomorphisms of M preserving the metric are the local isometries of (M,g). In general the semiriemannian space (M,g) is not locally homogeneous.

More general, if $(E,g) \to M$ is a semiriemannian vector bundle related to TM by a soldering form, it is this form that adds an extra structure besides the one of semiriemannian vector bundle, and therefore it defines a subsheaf of $O(E,g)$, which usually is not locally homogeneous.

Example A.29. Let $E \to M$ be a vector bundle of real dimension $2n, n \in \mathbb{N}^*$ over a differentiable manifold M. A $(1,1)$ tensor field J on M satisfying $J^2 = -I_{2n}$ defines an almost complex structure on the bundle $E \to M$. A riemannian metric h on E such that J is h-orthogonal is a hermitian metric on (E,J). We can always find h-unitary frames of vector fields in an appropriate neighborhood of any point of M, and therefore the $(1,1)$ h-unitary tensor fields from $E \otimes E^*$ form a locally homogeneous sheaf.

Example A.30. Let $M = \mathbb{R}$ be the base space. We can start with the set of all the polynomial functions of degree ≤ 1, $P_1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | (\exists a, b \in \mathbb{R})(\forall x \in \mathbb{R}) f(x) = ax + b\}$. Let the sheaf $\mathcal{P}_1(\mathbb{R})$ be generated by restrictions of $P_1(\mathbb{R})$. Considering two points $x_0, x_1 \in \mathbb{R}$, the translation $t_{x_1 - x_0} : \mathbb{R} \to \mathbb{R}$, $t_{x_1 - x_0}(x) = x + x_1 - x_0$, takes x_0 to x_1. If $f(x) = ax + b$, we have $(t_{x_1 - x_0} \circ f)(x) = ax + b + x_1 - x_0$, and therefore our sheaf $\mathcal{P}_1(\mathbb{R})$ is locally homogeneous.

Example A.31. As an example of inhomogeneous sheaf we can take $\mathcal{P}_1(\mathbb{R})_{(x_0,y_0)}$, the sheaf of all sections in $\mathcal{P}_1(\mathbb{R})$ above such that each one is a restriction of a function $\tilde{f} \in P_1(\mathbb{R})$ such that $\tilde{f}(x_0) = y_0$ for a given $(x_0, y_0) \in \mathbb{R}^2$. This sheaf is no longer locally homogeneous, because if we translate a line containing (x_0, y_0), in general we obtain a section of $\mathcal{P}_1(\mathbb{R})$ which does not belong to the sheaf $\mathcal{P}_1(\mathbb{R})_{(x_0,y_0)}$. The translation of the form $t_{x_1 - x_0}$ in fact associates to sections of $\mathcal{P}_1(\mathbb{R})_{(x_0,y_0)}$ sections of $\mathcal{P}_1(\mathbb{R})_{(x_1,y_1)}$. Even if we consider more general transformations from x_0 to another point, the only sections of $\mathcal{P}_1(\mathbb{R})_{(x_0,y_0)}$ which remains in $\mathcal{P}_1(\mathbb{R})_{(x_0,y_0)}$ are the constant ones.

Example A.32. Examples of semiriemannian manifolds (M,g) which are homogeneous: euclidean spaces, spheres and hyperbolic spaces.

A.4. **Intersection of subsheaves and homogeneity**

Let's consider two subsheaves of a sheaf. If they are locally homogeneous, is their intersection locally homogeneous too? Not necessarily, as we can see from the following example.

Example A.33. Let $E \to M$ be a vector bundle of real dimension $2n, n \in \mathbb{N}^*$ over a differentiable manifold M. Let J be a $(1,1)$-tensor providing a complex structure on E, and h a riemannian metric on E. Then the set of $(1,1)$-tensors leaving J invariant forms a sheaf $GL(E,J)$, and the set of h-orthogonal $(1,1)$-tensors forms a sheaf $O(E,h)$. If J itself is h-orthogonal, then the $(1,1)$-tensors of the sheaf $U(E,J,h) = GL(E,J) \cap O(E,h)$ are hermitian with respect to J and h, and the sheaf $\mathcal{H}$ is locally homogeneous. If J is not h-orthogonal, then the sheaf $\mathcal{I} = GL(E,J) \cap O(E,h)$ is not locally homogeneous. This happens because the intersection of the groups $GL(\mathbb{R}^{2n}, J)$ and $O(\mathbb{R}^{2n}, h)$ as subgroups of $GL(\mathbb{R}^{2n})$ at a point $x \in M$ where J is h-orthogonal, is isomorphic to the unitary group $U(n)$, while in the points where this condition is not satisfied it is not isomorphic to $U(n)$.

If, by contrary, two subsheaves are inhomogeneous, does this guarantee that their intersection is inhomogeneous too?

Example A.34. If we intersect two versions of the sheaf from the example A.31, $\mathcal{P}_1(\mathbb{R})_{(x_0,y_0)}$ and $\mathcal{P}_1(\mathbb{R})_{(x_1,y_1)}$, with $x_0 \neq x_1$, we obtain a locally homogeneous sheaf if $y_0 = y_1$, and an inhomogeneous sheaf if $y_0 \neq y_1$.

Appendix B. **Sheaf selections**

B.1. **Introduction**

If we consider a PDE system, a solution can be determined by a set of initial/boundary conditions. In general, considering a locally homogeneous sheaf, how can we impose conditions like the initial/boundary ones from the case of a PDE system, to determine a global section of the sheaf? In this section, I will define the *sheaf selections*, that translates into the sheaf language the initial/boundary conditions from the case of the PDE.

The main interest of the following is to analyze ways of specifying a section or a subsheaf of a sheaf, taken in general as locally homogeneous.

In order to discuss about subsheaves and sections, we will consider the sheaves as being contravariant functors between the category of open sets of a topological space and a category admitting subobjects (*e.g.* a concrete category). We will consider in general that our sheaf admits subsheaves.

Definition B.1. Let the sheaf $\mathcal{F}$ be a sheaf of sets. A subsheaf $\int \leq \mathcal{F}$ is said to be *section-like* if for any $U \in \mathcal{O}(\mathcal{S})$, $\int(U)$ has exactly one element.

Remark B.2. Any global section of $\mathcal{F}$ defines canonically a section-like subsheaf, provided that any extra structure on the sets $\mathcal{F}(U)$ is forgotten (in the sense of Category Theory). For example, if $\mathcal{F}$ is a sheaf of abelian groups, or a sheaf of modules, then the section-like subsheaves will be allowed to be subsheaves of the sheaf $\mathcal{F}$ considered as sheaf of sets. Reciprocally, each section-like subsheaf of $\mathcal{F}$ admits a unique global section, which is, of course, also a section of $\mathcal{F}$. Thus, there is a one-to-one correspondence between the global sections and the section-like subsheaves of a sheaf of sets, $\mathcal{F}$.

A global section in a sheaf of sets determines, via restrictions, a subsheaf, so it can be regarded as a particular type of subsheaf, namely a sheaf that associates to any non-empty open set of X a terminal object. This allows our discussion about sheaf selections to apply also to the selection of a global section of the sheaf.

B.2. **Sheaf selections**

Let's consider a condition imposed on the sections of a sheaf $\mathcal{F}$, resulting in a subsheaf $\mathcal{F}_0$. We can then forget about the condition, and consider only the subsheaf $\mathcal{F}_0$. If we want to determine a subsheaf $\mathcal{F}_0$ by a conjunction of such conditions, it is enough to consider an intersection of the subsheaves of $\mathcal{F}$ corresponding to each condition, their intersection will be $\mathcal{F}_0$. Each subsheaf participating to the intersection will be named *selector* of $\mathcal{F}_0$, and the set of sheaves whose intersection is $\mathcal{F}_0$ will be named *selection.*

We can therefore eliminate from our description the conditions imposed on the sections of the sheaf of $\mathcal{F}$ to obtain the subsheaf of $\mathcal{F}_0$, by replacing them with selections.

Let $\mathcal{F}$ be a sheaf over a topological space M. The set of all the subsheaves of the sheaf $\mathcal{F}$ form a lattice $Sub(\mathcal{F})$ with the partial order relation $\leq$.

Definition B.3. Let $\mathcal{F}$ be a sheaf over a topological space M, and $\mathcal{F}_0 \leq \mathcal{F}' \leq \mathcal{F}$ subsheaves. We say that the subsheaf $\mathcal{F}'$ is a *selector* of the subsheaf $\mathcal{F}_0$. Let σ be a collection of subsheaves of $\mathcal{F}$. The restriction of the partial order relation $\leq$ of subsheaves to σ is again a partial order. σ is named *selection* of the subsheaf $\mathcal{F}_0 \leq \mathcal{F}$ if

(1) All $\mathcal{F}' \in \sigma$ are selectors of $\mathcal{F}_0$.
(2) In the lattice $Sub(\mathcal{F})$, $\mathcal{F}_0 = \inf(\sigma)$.

Remark B.4.

(1) It follows from the definition that $(\sigma \cup \{\mathcal{F}_0\}, \leq)$ is a filtered set, with $\mathcal{F}_0$ as lower bound.
(2) A selector can be viewed as a (necessary) condition imposed on the sheaf $\mathcal{F}$. A selection is a sufficient set of conditions for $\mathcal{F}_0$.

Definition B.5. A selection σ is said to be *latticeal* if it is a lattice. If it is a totally ordered set we say that it is *totally ordered selection* or a *chain selection.* If σ is a sequence $\ldots \leq \mathcal{F} \leq \mathcal{F}' \ldots$, then it is named a *sequential selection.*

B.3. **Homogeneous sheaf selections**

We discussed about the locally homogeneous sheaves in the section A. Here I will provide a few examples of homogeneous subsheaf selections.

Example B.6. Let M be an analytical manifold. It's structure can be specified by a pseudogroup of analytical transformations, composed by the transition maps. We can consider a hierarchy $\mathcal{T}^{\omega}(M) \subset \mathcal{T}^{\infty}(M) \subset \ldots \subset \mathcal{T}^{k}(M) \subset \ldots \subset \mathcal{T}^{1}(M) \subset \mathcal{T}^{0}(M)$ of pseudogroups of transformations, from analytical transformations to differentiable transformations of finite degree, ending with continuous ones. These layers reflects the differentiable structures and the topological structure of an analytical manifold. Considering the real-valued functions on M, we obtain the following hierarchy of sheaves: $\mathcal{C}^{\omega}(M) \subset \mathcal{C}^{\infty}(M) \subset \ldots \subset \mathcal{C}^{k}(M) \subset \ldots \subset \mathcal{C}^{1}(M) \subset \mathcal{C}^{0}(M)$.

In the previous example the selection lattice was sequential. The following example provides us a latticeal selection which is not a sequential one.

Example B.7. Let V be an topological real vector space. The continuous linear forms on V define a sheaf which can be obtained by intersecting the sheaf obtained from the linear forms on V with the sheaf of continuous functions from V to $\mathbb{R}$. If the dimension is finite, the linear forms on V are also continuous, and the selection is sequential. In the infinite dimensional case, the linear forms on V are not necessarily continuous, therefore the selection is latticeal but no sequential.

Example B.8. An example of totally ordered selection which is not sequential, we can take the Sobolev spaces $\{W^s(X,E)\}_{s\in\mathbb{R}}$, of a Hermitian differentiable vector bundle $E \to X$. $\{W^s(X,E)\}_{s\in\mathbb{R}}$ satisfies $W^s \subset W^t$ for any $s > t$, and it is a selection of $W^{\infty}(X,E)$.

B.4. The locally homogeneous sheaf + selection decomposition

B.4.1. The first problem: selecting the solution

The first problem we want to express is the following. Considering a locally homogeneous sheaf $\mathcal{H}$, any global section of it can be expressed as a selection σ. This problem is similar to the problem of selecting a solution of a PDE subject to some initial/boundary conditions.

Let's consider $m+1$ $\mathbb{K}$-vector bundles $E_i \to M, i \in \{1,2,\ldots,m\}$, and $E \to M$ over a differentiable manifold M. By $\mathcal{E}(M,E)$ we will denote the ∞-differentiable sections of $E \to M$. Let $D_i : \mathcal{E}(M,E) \to \mathcal{E}(M,E_i)$ be m differential operators. By a *differential equation* we will understand an equation of the form

$$F(\xi, D_1(\xi), \ldots, D_m(\xi)) = 0, \tag{9}$$

where $F : E \otimes (\bigotimes_{i=1}^{m} E_m) \to \mathbb{K}$ is differentiable.

The solutions of the equation (9), when they exist, form a subsheaf $\mathcal{F}$ of the sheaf $\mathcal{E}(M,E)$. If the differentiable manifold M is homogeneous, so is the sheaf $\mathcal{F}$. It is possible to have local solutions, without having global ones.

In general the solution is not unique, and we need additional conditions to select it from the sheaf $\mathcal{F}$. Such conditions are in general of the form

$$\begin{cases} \xi|_{M_0} = \xi_0 \\ \tilde{D}_j \xi|_{M_0} = \chi_j, j \in \{1,\ldots,m'\} \end{cases} \tag{10}$$

where $\tilde{D}_j : \mathcal{E}(M,E) \to \mathcal{E}(M,F_j), j \in \{1,\ldots,m'\}$ are differential operators, $F_j \to M, j \in \{1,\ldots,m'\}$ vector bundles over M, $M_0 \overset{i}{\hookrightarrow} M$ is a submanifold of M (in general of codimension 1), ξ_0 is a section of i^*E, $\chi_j, j \in \{1,\ldots,m'\}$ are sections of i^*F_j.

We can see that the equation (9) describes a locally homogeneous sheaf, while the conditions (10) describe a selection.

B.4.2. **The second problem: the minimal sheaf description**

Let $\mathcal{F}_0$ be a subsheaf of a locally homogeneous sheaf $\mathcal{F}$. Then $\mathcal{F}_0$ can be expressed as the intersection of one homogeneous subsheaf $\mathcal{H}$ and one inhomogeneous subsheaf $\mathcal{I}$ of $\mathcal{F}$. Obviously, the subsheaves $\mathcal{H}$ and $\mathcal{I}$ are not unique. If $\mathcal{F}_0$ is a subsheaf generated by a global section s of $\mathcal{F}$, we can say that the section s is determined by $\mathcal{H}$ and $\mathcal{I}$.

Let $\sigma^{\mathcal{H}}$ be a selection of the subsheaf $\mathcal{H}$ and $\sigma^{\mathcal{I}}$ a selection of $\mathcal{I}$. The selection $\sigma = \sigma^{\mathcal{H}} \cup \sigma^{\mathcal{I}}$ is a selection of $\mathcal{F}_0$.

Let's start with a subsheaf $\mathcal{F}_0$ of a locally homogeneous sheaf $\mathcal{F}$. The second problem we present is to find a minimal locally homogeneous sheaf $\mathcal{H}$, $\mathcal{F}_0 \leq \mathcal{H} \leq \mathcal{F}$. In this way, $\mathcal{F}_0$ will be determined by a inhomogeneous selection $\sigma^{\mathcal{I}}$, together with $\mathcal{H}$. If by any chance there may exist locally homogeneous selectors in $\sigma^{\mathcal{I}}$, this would not necessarily mean that $\mathcal{H}$ is not minimal, as the Example A.33 shows.

A global section of a locally homogeneous sheaf can be locally homogeneous, by this meaning that the restriction mappings determine a locally homogeneous sheaf. One obvious example is when the section is constant. We don't impose to the selection $\sigma^{\mathcal{I}}$ to be inhomogeneous in the first problem, but only in the second one, when we are looking for a minimal description of $\mathcal{F}_0$.

Appendix C. **Homogeneity and selection on localic sheaves**

We begin this section by recalling some aspects of the Topos Theory. Then, we will extend the idea of local homogeneity to locales.

C.1. **Topos theoretical aspects**

We recall now a generalization of the sheaves on topological spaces. First we remember the definitions of a frame and of a locale (we are using [21, 18]):

Definition C.1. A *frame* is a lattice with all finite meets and all arbitrary (finite or infinite) joins, satisfying the infinite distributive law:

$$U \wedge (\bigvee_i U_i) = \bigvee_i (U \wedge U_i)$$

for any element U and any family of elements U_i.

Definition C.2. The frames can be considered as the objects of a category, which we name (Frames), having as morphisms maps of partially ordered sets which preserves the frame structure. This means that they preserves the finite meets and arbitrary joins. The objects of the dual category (Locales) := (Frames)^{op} are named *locales*. We denote the corresponding frame of a locale X by $\mathcal{O}(X)$.

For example, a topology is a locale. This fact allows us to define a functor

$$\mathrm{Loc} : (\mathrm{Spaces}) \to (\mathrm{Locales})$$

which associates to each topological space T the locale dual to the frame of its open subsets, by $\mathcal{O}(\mathrm{Loc}(T)) := \mathcal{O}(T)$.

Definition C.3. A *point* of a locale X is a map of locales $1 \to X$ from the terminal object 1 of the category (Locales) to X. We denote by $pt(X)$ the set of all points of the locale X. It is a topological space in a canonical manner, with the open sets $pt(U) = \{p \in pt(X) | p^{-1}(U) = 1\} \subseteq pt(X)$.

Definition C.4. If in a locale X we have $U = \bigvee U_i$, we say that $\{U_i\}_{i\in I}$ covers U. A *sheaf* on a locale X is a contravariant functor $\mathcal{F} : \mathcal{O}(X) \to Set$ such that $(r_{U_i})_{i\in I}$ (the restriction maps) is an equalizer for the diagram

$$\mathcal{F}(U) \xrightarrow{(r_{U_i})_{i\in I}} \prod_{i\in I} \mathcal{F}(U_i) \underset{(f_i)_{i\in I} \mapsto (f_j|_{U_i\cap U_j})_{i,j\in I}}{\overset{(f_i)_{i\in I} \mapsto (f_i|_{U_i\cap U_j})_{i,j\in I}}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

for any open covering $U = \bigvee_{i\in I} U_i$ of an $U \in \mathcal{O}(X)$.

Definition C.5. A *topos* is a Cartesian closed category with a subobject classifier. An equivalent definition is that a topos is a finitely complete and finitely co-complete category with exponentiation and subobject classifier. A *localic topos* is a topos equivalent to $Sh(X)$, for a locale X.

Definition C.6. If $F : A \to B$ and $G : B \to A$ are two functors such that there is a family of bijections $Hom_B(FX, Y) \cong Hom_A(X, GY)$, for any pair $X \in A$ and $Y \in B$, which is natural in X and Y, then F is said to be *left adjoint* to G, and G *right adjoint* to F.

Definition C.7. A functor $F : A \to B$ is said to be *exact* if for any short exact sequence $0 \to X \to Y \to Z \to 0$, the sequence $0 \to FX \to FY \to FZ \to 0$ is exact. If only $FX \to FY \to FZ$ ($0 \to FX \to FY \to FZ$, $FX \to FY \to FZ \to 0$) is exact then the functor F is named *half* (*left*, *right*) *exact*.

Definition C.8. A *geometric morphism* between two topoi $f : \mathcal{E} \to \mathcal{E}'$ consists in a left exact functor $f^* : \mathcal{E}' \to \mathcal{E}$ (named the *inverse image* part of f) which is a left adjoint to a functor $f_* : \mathcal{E} \to \mathcal{E}'$ (named the *direct image* part of f).

By a result of Diaconescu ([14, 15]), from any topos $\mathcal{F}$ we can construct a locale X, and then a geometric morphism $\mathrm{Sh}(X) \to \mathcal{F}$.

C.2. **Local homoegeneity on locales**

Definition C.9. Let X be a locale. If $U \in \mathcal{O}(X)$, then the comma category $\mathcal{O}(X) \downarrow U$ is a frame and defines a locale which will be denoted by $X \downarrow U$, and which is said to be a *comma locale* of X. $X \downarrow U$ is isomorphic to a full subcategory of X.

Definition C.10. By a *transformation* of a locale X we understand a category isomorphism $F : X \cong X$. A *local transformation* of the locale X will be an isomorphism between two comma locales of X. The category of all the comma locales of X, having as morphisms the local transformations, form the *category of local transformations* of X, $\mathcal{T}(X)$.

Definition C.11. We say that a locale X is *locally homogeneous* if for any two points $p, q : 1 \to X$ there exist $U, V \in \mathcal{O}(X)$ and a local transformation $T : U \to V$ such that the points $p \in pt(U)$ and $q \in pt(V)$, and $q = T \circ p$.

Definition C.12. A *category of local transformations* on a locale X is a category $\mathcal{T}$ satisfying:

(1) $Ob(\mathcal{T}) = Ob(X)$.
(2) X is a full subcategory of $\mathcal{T}$. This is to say that the identities 1_U for any $U \in Ob(X)$ and the inclusions $U \hookrightarrow V$ for any U, V with $U \wedge V = U$ are also morphisms of $\mathcal{T}$.
(3) If $T : U \to V$ is a transformation in $\mathcal{T}$ and $U' < U$, then the restriction $T|_{U'}$ of T to $U \downarrow U'$ is also in $\mathcal{T}$.
(4) If $T : U \to V$ is a transformation (not necessarily in $\mathcal{T}$), $U = \vee U_i$, where $U_i \in Ob(\mathcal{T})$, and $T|_{U_i} \in Hom(\mathcal{T})$ for any $i \in I$, then $T \in Hom(\mathcal{T})$.

Definition C.13. A sheaf $\mathcal{F}$ on a locally homogeneous locale is said to be *locally homogeneous* if for any two points $p, q \in pt(X)$ there is a local transformation $T : U \to V$ such that $q = T(p)$ and $T(\mathcal{F}(U)) = \mathcal{F}(V)$.

The local transformations of a locale X preserving the sheaf structure of a sheaf $\mathcal{F}$ on X form a category of transformations on X, which we denote by $\mathcal{T}(\mathcal{F})$.

The concept of sheaf selection can be extended straightforwardly to sheaves on locales.

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